ORLICZ FUNCTION SPACES WITHOUT COMPLEMENTED COPIES OF l^p †

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ABSTRACT
This paper proves the existence of Orlicz function spaces $L^\Phi(0,1)$ containing no complemented subspaces isomorphic to l^p for any $p \neq 2$. Some properties of minimal Orlicz function spaces $L^\Phi(0,1)$ are also given.

The existence of Orlicz sequence spaces l^ϕ containing no complemented subspaces isomorphic to l^p for any $p \geqq 1$ was proved by J. Lindenstrauss and L. Tzafriri ([2], [3], [4]) by introducing the important class of minimal Orlicz sequence spaces l^ϕ.

In this note we show a corresponding result for Orlicz function spaces $L^\Phi(0,1)$. We consider minimal Orlicz function spaces $L^\Phi(0,1)$ in order to prove the existence of Orlicz spaces $L^\Phi(0,1)$ that contain no complemented subspaces isomorphic to l^p for any $p \neq 2$. More precisely the following result will be proved:

THEOREM. *Given $1 < r \leqq s \leqq 2$ or $2 \leqq r \leqq s < \infty$, there exists an Orlicz function space $L^\Phi(0,1)$ with indices $\alpha_\Phi^\infty = r$ and $\beta_\Phi^\infty = s$ which contains no complemented subspaces isomorphic to l^p for any $p \neq 2$.*

First let us recall some definitions. If ϕ is an Orlicz function (i.e., a continuous convex non-decreasing function defined for $x \geqq 0$ such that $\phi(0) = 0$ and $\phi(1) = 1$) and μ is the Lebesgue measure on $[0,1]$, the *Orlicz space* $L^\Phi(0,1) \equiv L^\Phi$ consists of all measurable functions f on $[0,1]$ such that

$$m_r(f) = \int_0^1 \phi\left(\frac{|f|}{r}\right) d\mu < \infty \qquad \text{for some } r > 0.$$

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The space L^ϕ endowed with the Luxemburg norm $\|f\| = \inf\{r > 0\colon m_r(f) \leqq 1\}$ is a Banach space. We assume that ϕ satisfies the Δ_2-condition, so the space L^ϕ is separable ([6], [5]).

We shall consider the following sets related to ϕ in the space $C(0,\infty)$ endowed with the compact-open topology:

$$E_{\phi,s} = \overline{\left\{\frac{\phi(rt)}{\phi(r)}: r \leqq s\right\}}, \qquad E_\phi = \bigcap_{s>0} E_{\phi,s},$$

$$E^\infty_{\phi,s} = \overline{\left\{\frac{\phi(rt)}{\phi(r)}: r \geqq s\right\}}, \qquad E^\infty_\phi = \bigcap_{s>0} E^\infty_{\phi,s},$$

$$C_{\phi,s} = \overline{\operatorname{conv}} E_{\phi,s}, \qquad C_\phi = \overline{\operatorname{conv}} E_\phi,$$

for every $s > 0$. As in the case of $C(0,1)$ (see [2], [4]) it follows from the Δ_2-condition that the above sets are compact subsets of the space $C(0,\infty)$.

Let us define now a concept of "minimality" in Orlicz spaces which extends the one given by J. Lindenstrauss and L. Tzafriri in the context of Orlicz sequence spaces l^ϕ ([2], [3])

Definition. An Orlicz function ϕ is *minimal* at ∞ (resp. at 0) if for every function $\psi \in E^\infty_{\phi,1} \subset C(0,\infty)$ (resp. $E_{\phi,1} \subset C(0,\infty)$) we have that $E^\infty_{\phi,1} = E^\infty_{\psi,1}$ (resp. $E_{\phi,1} = E_{\psi,1}$).

The existence of minimal functions at ∞ (resp. at 0) is proved by Zorn's Lemma as in ([2], [4]).

Proposition 1. *Let ϕ be a minimal function at ∞ (resp. at 0). Then in $C(0,\infty)$*

$$E^\infty_{\phi,1} = E^\infty_\phi = E_\phi = E_{\phi,1}.$$

Proof. Let us assume that ϕ is minimal at ∞. If $\psi \in E^\infty_\phi \neq \phi$ it is clear that $E^\infty_{\psi,1} \subset E^\infty_\phi$. Since $E^\infty_{\phi,1} = E^\infty_{\psi,1}$ we deduce that $E^\infty_{\phi,1} = E^\infty_\phi$. Hence there exists a sequence $(r_n) \uparrow \infty$ such that $\phi(r_n\,\cdot)/\phi(r_n)$ converges to ϕ uniformly over the compact sets. Then the functions

$$\frac{\phi(r_n r\,\cdot)}{\phi(r_n r)} = \frac{\phi(r_n r\,\cdot)}{\phi(r_n)} \cdot \frac{\phi(r_n)}{\phi(r_n r)}$$

converge pointwise to $\phi(r\,\cdot)/\phi(r)$ for every $r \in (0,\infty)$. As $(r_n r) \uparrow \infty$, we deduce that

$$\overline{\left\{\frac{\phi(r\,\cdot)}{\phi(r)}: r \leqq 1\right\}} = E_{\phi,1} \subset E^\infty_\phi.$$

Now we can take in $E_{\phi,1}$ a minimal function M at 0 and, by the same arguments as above, exchanging the roles of 0 and ∞, we obtain that $E_{M,1} = E_M$ and $E^{\infty}_{M,1} \subset E_M$.

Finally, we have

$$E_{\phi,1} \supset E_\phi \supset E_M = E_{M,1} \supset E^{\infty}_{M,1} = E^{\infty}_{\phi,1} = E^{\infty}_{\phi} \supset E_{\phi,1}. \qquad \text{Q.E.D.}$$

As a consequence we get that a function ϕ is minimal at ∞ if and only if it is minimal at 0. From now on we shall say, in short, *minimal functions.* It is also deduced easily that a minimal function has the same indices at ∞ and at 0, i.e., $\alpha^{\infty}_{\phi} = \alpha_\phi$ and $\beta^{\infty}_{\phi} = \beta_\phi$ (as defined in [4], [5]).

Let us remark now on the relation with the Lindenstrauss–Tzafriri (L–T) minimal sequence spaces l^ϕ. Obviously every minimal function is a L–T minimal function ([2]). Conversely, if ϕ is a L–T minimal function then the restriction to $[0,1]$ can be extended over the whole $[0,\infty)$ defining a function M which is minimal in $C(0,\infty)$. Indeed, in $E_{\phi,1} \subset C(0,\infty)$ there exists a minimal function ψ. Now, since ψ is also a L–T minimal function, we can take a sequence $(\psi(r_k\,\cdot)/\psi(r_k))$ in $E_{\psi,1}$ which converges to $\phi_{|[0,1]}$ in $C(0,1)$ and also to a function M in $C(0,\infty)$. Hence the function M is minimal and $M_{|[0,1]} = \phi$.

PROPOSITION 2. *Let ϕ be a minimal Orlicz function. Then the Orlicz function space L^ϕ has a complemented subspace isomorphic to l^ϕ.*

PROOF. Let us consider the Orlicz sequence spaces $l^\phi(w)$ defined by

$$l^\phi(w) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \phi\left(\frac{|x_n|}{s}\right) w_n < \infty \text{ for some } s > 0 \right\}$$

where (w_n) is an arbitrary sequence of positive scalars ([1]). It is clear that the space L^ϕ has a complemented copy of $l^\phi(w)$ for $\sum_{n=1}^{\infty} w_n < \infty$, by considering a conditional expectation.

Now let us see that l^ϕ is isomorphic to any space $l^\phi(w)$ for sequences $(w_n)_1^\infty$ of finite sum. Indeed, if $(r_n)_{n=1}^{\infty}$ denotes the scalar sequence verifying $1/\phi(r_n) = w_n$, then the functions $\phi(r_n\,\cdot)/\phi(r_n)$ belong to $E^{\infty}_{\phi,1}$ for sufficiently large n. Since $E^{\infty}_{\phi,1} = E_\phi$ we can take a sequence $(s_n)_{n=n_0}^{\infty}$ converging to 0 such that

$$\left| \frac{\phi(r_n t)}{\phi(r_n)} - \frac{\phi(s_n t)}{\phi(s_n)} \right| \leq \frac{1}{2^n} \qquad \text{for } 0 \leq t \leq 1$$

and sufficiently large n. Now for $w'_n = 1/\phi(s_n)$, it follows from the above relation that the spaces $l^\phi(w)$ and $l^\phi(w')$ are isomorphic. Finally, as $w'_n \to \infty$ the space $l^\phi(w')$ is isomorphic to a space generated by a block basis with constant coefficients of the unit vector basis of l^Φ. So from the minimality of the space l^Φ ([4] Prop. 4.b.7) we conclude $l^\Phi(w') \approx l^\Phi$. Q.E.D.

PROPOSITION 3. *Let ϕ be a minimal Orlicz function. If L^Φ has an isomorphic copy of an Orlicz sequence space l^ψ with $\beta_\psi > 2$, then there exists in $C_{\psi,1}$ a function F equivalent at 0 to some function of $C_{\phi,1}$.*

PROOF. Let Y be a subspace of L^Φ isomorphic to l^ψ. Since Y contains a copy of l^p for $p = \beta_\psi > 2$ ([4]) and L^1 has cotype 2, we have that Y is not isomorphic to any subspace of L^1. Now we will apply the generalized Kadec–Pelczynski method ([5] Prop. 1.c.8): Let us consider the sets

$$\sigma(f,\varepsilon) = \{t : |f(t)| \geqq \varepsilon \|f\|\} \quad \text{and} \quad M(\varepsilon) = \{f \in L^\Phi : \mu(\sigma(f,\varepsilon)) \geqq \varepsilon\}$$

for $f \in L^\Phi$ and $\varepsilon > 0$. For each $n > 2$, there exists an $f_n \in Y$ with $\|f_n\| = 1$ and $f_n \notin M(1/2^n)$. If T is the isomorphism between Y and l^ψ, then $(T(e_i))_1^\infty$ is a basis of Y and we can choose functions $u_n = \sum_{i=1}^{N_n} a_i T(e_i)$ of Y verifying that

$$1 - \frac{1}{2^n} \leqq \|u_n\| \leqq 1 + \frac{1}{2^n} \quad \text{and} \quad |u_n(t) - f_n(t)| < \frac{1}{2^n}$$

outside of a set $A_n \subset [0,1]$ of measure $\mu(A_n) < 1/2^n$.

We claim that $u_n \notin M(1/2^{n-2})$. Indeed, it is clear that

$$\sigma\left(u_n, \frac{1}{2^{n-2}}\right) \subset \left\{t : |u_n(t)| > \frac{3}{2^n}\right\} = B_n.$$

Now if $t \in B_n \setminus A_n$ then $|f_n(t)| \geqq 1/2^{n-1} > 1/2^n$, so $t \in \sigma(f_n, 1/2^n)$. Hence

$$\mu\left(\sigma\left(u_n, \frac{1}{2^{n-2}}\right)\right) \leqq \mu(B_n) < \frac{1}{2^n} + \frac{1}{2^n} < \frac{1}{2^{n-2}}$$

and $u_n \notin M(1/2^{n-2})$.

Furthermore, in the above construction we can replace in each step corresponding to $n > 3$ the subspace Y by the subspace $Y_n = \overline{[T(e_i)_{i=N_{(n-1)}}^\infty]}$, which is isomorphic to Y since the basis $(T(e_i))_1^\infty$ is subsymmetric. Hence we obtain a block basis of $(T(e_i))_1^\infty$ that we still denote by $(u_n)_1^\infty$, and by a routine argument ([4], p. 142) there exists a subsequence $(u_{n_k})_{k=1}^\infty$ of $(u_n)_1^\infty$ generating in Y an Orlicz sequence space l^F for some function $F \in C_{\psi,1}$.

Now, working with a subsequence $(u_{n_{k_j}})_{j=1}^\infty$ of $(u_{n_k})_{k=1}^\infty$ as in the proof of ([5] Prop. 1.c.8), we can find functions $(g_j)_{j=1}^\infty$ of L^Φ with mutually disjoint supports verifying

$$\|g_j - u_{n_{k_j}}\| < \frac{1}{2^{j-1}} \qquad \text{for } j = 1, 2, \ldots .$$

Hence, by a perturbation result ([4] Prop. 1.a.9i), the basic sequences $(g_j)_1^\infty$ and $(u_{n_{k_j}})_{j=1}^\infty$ are equivalent, so the subspace $\overline{[(g_j)]}_{j=1}^\infty$ is isomorphic to $\overline{[(u_{n_{k_j}})]}_{j=1}^\infty \approx l^F$.

Finally, by the density of the step functions in L^Φ, for each j there exist mutually disjoint sets $B_{j,r} \subset \operatorname{supp}(g_j)$ and real numbers $(a_{j,r})$ for $r = 1, \ldots, k_j$ such that $h_j = \sum_{r=1}^{k_j} a_{j,r}\chi_{B_{j,r}}$ verifies

$$\| g_j - h_j \| < 1/2^j.$$

Hence the space $\overline{[(h_j)]}_{i=1}^\infty$ is isomorphic to l^F. On the other hand, the subspace $\overline{[(\chi_{B_{j,r}})]}_{j,r}$ is isomorphic to l^Φ by the proof of Proposition 2. Therefore l^Φ contains a subspace isomorphic to l^F and from ([4] Thm. 4.a.8) we conclude that F is equivalent at 0 to a function of $C_{\Phi,1}$. Q.E.D.

Proposition 4. *Let ϕ be a minimal Orlicz function and $p > 2$. Then L^Φ has a copy (resp. a complemented copy) of l^p if and only if l^Φ has a copy (resp. a complemented copy) of l^p.*

Proof. One of the implications is a simple consequence of Proposition 2. Now if L^Φ has a copy of l^p, we apply Proposition 3 for $\psi(t) = t^p$ and get that l^Φ has a copy of l^p.

Let us assume now that L^Φ has a complemented subspace Y isomorphic to l^p. Repeating the proof of Proposition 3 we obtain, with the same notation, the block basis $(u_{n_k})_{k=1}^\infty$ of Y in L^Φ. Since every block basis of the canonical basis of l^p is complemented ([4] Prop. 2.a.1) we get that the space $\overline{[(u_{n_k})]}_{k=1}^\infty \approx l^p$ is complemented in Y and hence in L^Φ. Now, by taking an adequate perturbation ([4] Prop. 1.a.9ii), it is possible to obtain basic sequences $(g_j)_{j=1}^\infty$ and $(h_j)_{j=1}^\infty$ which are complemented in L^Φ. thus l^Φ has a complemented subspace isomorphic to $\overline{[(h_j)]}_{j=1}^\infty \approx l^p$. Q.E.D.

It is well known that in every reflexive Orlicz function space L^Φ the Rademacher functions span is a complemented subspace isomorphic to l^2 (e.g. [5]).

Proof of the Theorem. Fix $2 \leqq r \leqq s < \infty$; let us consider the minimal Orlicz function ϕ defined by Lindenstrauss and Tzafriri in ([3], [4] Example 4.c.7) with indices $2 \leqq r = \alpha_\Phi \leqq s = \beta_\Phi$. Thus the minimal Orlicz sequence spaces l^Φ do not have any complemented subspace isomorphic to l^p for $p \geqq 1$. Now, as we remarked above, the function ϕ on $[0,1]$ can be extended to a minimal function in $C(0,\infty)$ which we also denote by ϕ. Hence the indices of ϕ are $\alpha_\Phi^\infty = r$ and $\beta_\Phi^\infty = s$. Since $\alpha_\Phi^\infty \geqq 2$ it follows from ([3], p. 386) that L^Φ contains no copies of l^p for $p \not\in [\alpha_\Phi^\infty, \beta_\Phi^\infty] \cup \{2\}$. So, using the above Proposition, we conclude that the

Orlicz space L^{Φ} does not have complemented subspaces isomorphic to l^p for $p \neq 2$.

The remaining case is now easily proved by using duality arguments. Q.E.D.

REMARK. We do not know whether the above result is still true when $r < 2 < s$.

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